Received: *November 27, 2000*, Accepted: *May 2, 2001*
Revised: *March 20, 2001*

A note on compactifications on Spin(7)-holonomy manifolds

Katrin Becker

California Institute of Technology 452-48, Pasadena, CA 91125
CIT-USC Center for Theoretical Physics
E-mail: beckerk@theory.caltech.edu

Abstract: In this note we consider compactifications of $\mathcal{M}$-theory on Spin(7)-holonomy manifolds to three-dimensional Minkowski space. In these compactifications a warp factor is included. The conditions for unbroken $N = 1$ supersymmetry give rise to determining equations for the 4-form field strength in terms of the warp factor and the self-dual 4-form of the internal manifold.

Keywords: M-Theory, Superstring Vacua.

JHEP05(2001)003

Warped compactifications of $\mathcal{M}$-theory and $\mathcal{F}$-theory have attracted recently much attention in connection to confining gauge theories [1, 2, 3, 4, 5] and the string theoretic realization of the Randall-Sundrum scenario [6, 7], suggested in [8]. In [9] the conditions for unbroken supersymmetry for compactifications on Calabi-Yau 4-folds were found. These are manifolds that admit two covariantly constant spinors. It has recently been shown that the model considered by Klebanov and Strassler [3] can be obtained from the general type of solutions presented in [9].

From this perspective it is rather interesting to understand the physics of warped compactifications of $\mathcal{M}$-theory.[1] In this note we would like to describe compactifications of $\mathcal{M}$-theory on Spin(7)-holonomy manifolds. These are eight-manifolds that admit only one covariantly constant Majorana-Weyl spinor which arises from the the decomposition $8_c \to 7 \oplus 1$. Therefore these compactifications will give rise to $N = 1$ theories in three dimensions while the models considered in [9] had an $N = 2$ supersymmetry. These theories are interesting because they cannot be obtained from a compactification of any supersymmetric four-dimensional theory on an S^1. In fact, a theory with $N = 1$ in $d = 4$ would yield an $N = 2$ theory in three-dimensions [14].

Spin(7)-holonomy manifolds can be treated in a similar way as the Calabi-Yau 4-fold case considered in [9] so we will be brief here and use the notations and conventions of [9].

To derive the conditions following from unbroken supersymmetry we start with the supersymmetry transformation of the gravitino in eleven-dimensional supergravity

$$\delta\Psi_M = \nabla_M \eta - \frac{1}{288}(\Gamma_M{}^{PQRS} - 8\delta_M^P \Gamma^{QRS}) F_{PQRS}\eta = 0 \,. \tag{1}$$

We make the following ansatz for the metric

$$g_{MN}(x,y) = \Delta^{-1}(y) \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{pmatrix} , \tag{2}$$

where x are the coordinates of the external space labeled by the indices $\mu, \nu, \ldots$ and y are the coordinates of the internal manifold labeled by $m, n, \ldots$ $\Delta = \Delta(y)$ is the warp factor.

The eleven-dimensional spinor η is decomposed as

$$\eta = \epsilon \otimes \xi \,, \tag{3}$$

where ϵ is a three-dimensional anticommuting spinor and ξ is an eight-dimensional Majorana-Weyl spinor. Furthermore we make the decomposition of the gamma matrices

$$\begin{aligned} \Gamma_\mu &= \gamma_\mu \otimes \gamma_9 \,, \\ \Gamma_m &= 1 \otimes \gamma_m \,, \end{aligned} \tag{4}$$

[1]Warped compactifications on 4-folds have been further considered in [10, 11, 12] and [13].

JHEP05(2001)003

where γ_μ and γ_m are the gamma matrices of the external and internal space respectively. We choose the matrices γ_m to be real and antisymmetric. γ_9 is the eight-dimensional chirality operator which anti-commutes with all the γ_m's.

In compactifications with maximally symmetric three-dimensional space-time the non-vanishing components of the 4-form field strength F_4 are

$$\begin{aligned} &F_{mnpq} \quad \text{arbitrary} \\ &F_{\mu\nu\rho m} = \epsilon_{\mu\nu\rho} f_m \,, \end{aligned} \tag{5}$$

where F_{mnpq} and f_m will be determined from the conditions following from unbroken supersymmetry and $\epsilon_{\mu\nu\rho}$ is the Levi-Civita tensor of the three-dimensional external space. The external component of the gravitino supersymmetry transformation is given by the following expression

$$\begin{aligned} \delta\psi_\mu = \nabla_\mu \eta - \frac{1}{288}\Delta^{3/2}(\gamma_\mu \otimes \gamma^{mnpq})F_{mnpq}\eta + \\ + \frac{1}{6}\Delta^{3/2}(\gamma_\mu \otimes \gamma^m) f_m \eta + \frac{1}{4}\partial_n(\log\Delta)(\gamma_\mu \otimes \gamma^n)\eta \,, \end{aligned} \tag{6}$$

where we have used a positive chirality eigenstate $\gamma_9 \xi = \xi$. Considering negative chirality spinors corresponds to an eight-manifold with a reversed orientation [15]. Since we would like to compactify $\mathcal{M}$-theory to three-dimensional Minkowski space we impose the condition

$$\nabla_\mu \epsilon = 0 \,. \tag{7}$$

The external component of supersymmetry is then reduced to the equation

$$T\xi = 0 \qquad \text{with } T = F_{mnpq}\gamma^{mnpq} - 48(f_n - \partial_n \Delta^{-3/2})\gamma^n \,. \tag{8}$$

Taking into account that ξ is Weyl we conclude that the external component of F_4 can be expressed in terms of the warp factor

$$F_{\mu\nu\rho m} = \epsilon_{\mu\nu\rho}\partial_m \Delta^{-3/2} \,, \tag{9}$$

while the internal component of F_4 (which we denote by F) is constrained to satisfy

$$F_{mnpq}\gamma^{mnpq}\xi = 0 \,. \tag{10}$$

The analysis of the internal components of supersymmetry can be performed as in [9]. We find that in terms of the transformed quantities

$$\begin{aligned} \tilde{g}_{mn} &= \Delta^{-3/2} g_{mn} \,, \\ \tilde{\xi} &= \Delta^{1/4}\xi \,, \end{aligned} \tag{11}$$

the internal component of the gravitino transformation is given by

$$\tilde{\nabla}_m \tilde{\xi} + \frac{1}{24}\Delta^{-3/4} F_m \tilde{\xi} = 0 \,. \tag{12}$$

JHEP05(2001)003

The metric $\tilde{g}_{mn}$ describes the Spin(7)-holonomy manifold. These manifolds are Ricci flat and they admit one covariantly constant spinor which satisfies

$$\tilde{\nabla}_m \tilde{\xi} = 0\,. \tag{13}$$

Therefore we see that F has to satisfy

$$F_{mnpq} \tilde{\gamma}^{npq} \tilde{\xi} = 0\,. \tag{14}$$

Note that the condition (14) is actually stronger than (10).[2] However it can be shown that if F is self-dual then (14) is equivalent to (10).

The proof that F is self-dual goes as follows. From (14) we obtain the equation

$$F_{mnpq} \tilde{\xi}^T \{\tilde{\gamma}^{npq}, \tilde{\gamma}_{abc}\} \tilde{\xi} = 0\,. \tag{15}$$

To further evaluate (15) we note that we can construct covariantly constant p-forms in terms of the eight-dimensional spinor $\tilde{\xi}$

$$\omega_{a_1 \ldots a_p} = \tilde{\xi}^T \tilde{\gamma}_{a_1 \ldots a_p} \tilde{\xi}\,. \tag{16}$$

Since $\tilde{\xi}$ is Majorana-Weyl (16) is non-zero only for $p = 0, 4$ or 8 (see [17]). The Spin(7) calibration is then given by the closed self-dual 4-form

$$\Phi_{mnpq} = \tilde{\xi}^T \tilde{\gamma}_{mnpq} \tilde{\xi}\,. \tag{17}$$

If we would have considered negative chirality spinors this form would be anti-self-dual [18]. Taking this definition of the calibration into account and (15) we obtain

$$F_{mnpq} = \frac{3}{2} F_{abm[n} \Phi_{pq]}{}^{ab}\,. \tag{18}$$

By considering the quantity

$$F_{mnpq} \tilde{\xi}^T \{\tilde{\gamma}_{abcd}, \tilde{\gamma}^{mnpq}\} \tilde{\xi} = 0\,, \tag{19}$$

we can derive the condition

$$\star F_{mnpq} + F_{mnpq} = 3 F_{ab[mn} \Phi_{pq]}{}^{ab}\,, \tag{20}$$

where by $\star$ we mean the Hodge dual with respect to the metric of the eight-dimensional internal space. Antisymmetrizing the right hand side of (18) over the indices $(mnpq)$ and comparing with (20) we see that F satisfies the self-duality condition

$$F = \star F\,. \tag{21}$$

This self-duality condition can also be obtained from the equation of motion of F by using the explicit form of the external component of F_4 [16].

[2]This equation has been noticed before in [16].

JHEP05(2001)003

Taking this self-duality condition into account, we next would like to show that the constraint (14) coming from the internal component of supersymmetry is equivalent to the condition (10). For this purpose a useful identity to consider is

$$F_m F^m = -\frac{1}{8}\mathcal{F}^2 - 3F_{mnpq}\left(F^{mnpq} - \star F^{mnpq}\right), \tag{22}$$

where $\mathcal{F} = F_{mnpq}\tilde{\gamma}^{mnpq}$. Since F is self-dual the second term on the right hand side (22) vanishes. Equation (22) then implies that (14) is satisfied if and only if (10) is fulfilled.

To see the conditions imposed by (10) on our flux we note that Fierz rearrangements imply

$$F_{mnpq}\tilde{\gamma}^{mnpq}\tilde{\xi} = F_{mnpq}\Phi^{mnpq}\tilde{\xi} - F_{mnpr}\Phi_s{}^{mnp}\tilde{\gamma}^{rs}\tilde{\xi}\,. \tag{23}$$

The condition for unbroken supersymmetry states that the left hand side of (23) vanishes. After multiplying this equation by $\tilde{\xi}^T$ we get

$$F \wedge \Phi = 0\,. \tag{24}$$

Vanishing of the second term on the right hand side of (23) then implies

$$\omega_{rs}\tilde{\gamma}^{rs}\tilde{\xi} = 0\,, \tag{25}$$

where we have defined a 2-form ω as

$$\omega = \frac{1}{2}F_{mnpr}\Phi_s{}^{mnp}dx^r \wedge dx^s\,. \tag{26}$$

The spinors $\tilde{\gamma}^{rs}\tilde{\xi}$ are not independent [19]. To satisfy (25) ω has to obey the self-duality constraint

$$\frac{1}{2}\Phi^{rspq}\omega_{pq} = \lambda\omega^{rs}\,, \tag{27}$$

with $\lambda = 1$. But by taking the relation

$$\Phi^{mnpt}\Phi_{qrst} = 6\delta^{mnp}_{qrs} - 9\delta^{[m}_{[q}\Phi^{np]}{}_{rs]} \tag{28}$$

and the definition of ω into account it is easy to see that ω satisfies (27) with $\lambda = -3$. Therefore we conclude

$$\omega = 0\,. \tag{29}$$

Equations (21), (24) and (29) are the determining equations for F. They are the necessary and sufficient conditions for (14) to be satisfied.

After imposing the self-duality condition (20) takes the form

$$F_{mnpq} = \frac{3}{2}F_{ab[mn}\Phi_{pq]}{}^{ab}\,. \tag{30}$$

By contracting this equation with Φ^{npqr} it is possible to show that (30) is equivalent to (24) and (29). Therefore this is another way of expressing the condition for unbroken supersymmetry. In this form the condition for unbroken supersymmetry is similar to the one satisfied by Yang-Mills fields $\mathcal{F}_{mn}$ for which the following equivalence relation holds [20]

$$\mathcal{F}_{mn} = \frac{1}{2}\Phi_{mnpq}\mathcal{F}^{pq} \qquad \Longleftrightarrow \qquad \mathcal{F}_{mn}\tilde{\gamma}^{mn}\tilde{\xi} = 0\,. \tag{31}$$

The determining equation for the warp factor is the fivebrane Bianchi identity which after using our solution for F_4 takes the same form as in [9]

$$d\star d\log\Delta = \frac{1}{3}F\wedge F - \frac{2}{3}(2\pi)^4 X_8\,. \tag{32}$$

For compactifications of $\mathcal{M}$-theory on Spin(7)-holonomy manifolds we can expect to find non-vanishing expectation values for F_4 independently of the fact that the manifold is compact or not. This is different than the situation considered in [21] in which compactifications of $\mathcal{M}$-theory on seven-dimensional manifolds were considered and which only had non-vanishing expectation values for F in the case that the internal manifold was non-compact.

For a compact Spin(7)-holonomy manifold K^8 we can integrate (32) and obtain the relation

$$\int_{K^8} F\wedge F + \frac{1}{12}\chi = 0\,, \tag{33}$$

where χ is the Euler number of the eight-manifold [9, 22].

To summarize we have found the following conditions for unbroken supersymmetry for compactifications of $\mathcal{M}$-theory on manifolds with Spin(7) holonomy to three-dimensional Minkowski space: the internal components of F_4 obey the constraints (21), (24) and (29), the external components of F_4 are determined in terms of the warp factor by (9) and the warp factor can then be obtained from equation (32).

Concrete examples of compact Spin(7)-holonomy manifolds have been constructed in [23] (see also [24] for further discussion). A non-compact example was discussed recently in [25][3] and was constructed in [17]. In this case the explicit form of the metric is known and it takes the form of a quaternionic line bundle over a 4-sphere:

$$ds_8^2 = \alpha(r)^2 dr^2 + \beta(r)^2(\sigma^i - A^i)^2 + \gamma(r)^2 d\Omega_4^2\,, \tag{34}$$

where σ_i are left-invariant 1-forms of SU(2), A^i are SU(2) Yang-Mills potentials on the unit 4-sphere whose metric is $d\Omega_4^2$. We will be following the notation of [25] and refer the reader to this work for further details.

In [25] an explicit computation of F was done. In the following we would like to show that this solution satisfies our equations. This solution is anti-self-dual. This would correspond to choosing spinors with negative chirality (i.e. which satisfy $\gamma_9\tilde{\xi} = -\tilde{\xi}$) instead of the positive chirality spinors that we have used here. To show

[3] See also [16].

that equation (24) is satisfied we need the explicit forms of Φ and F. They are given by

$$\Phi = f_1' dr \wedge \epsilon_{(3)} - (f_1 + g_1) Y_{(4)} + g_1' dr \wedge X_3 - 6 g_1 \Omega_{(4)} \,, \tag{35}$$

with

$$f_1 = \frac{1}{5} c_1 (1 - 6z) z^{-6/5} \qquad \text{and} \qquad g_1 = c_1 z^{-6/5} \,, \tag{36}$$

while F is given by

$$F = f_2' dr \wedge \epsilon_{(3)} - (f_2 + g_2) Y_{(4)} + g_2' dr \wedge X_3 - 6 g_2 \Omega_{(4)} \,, \tag{37}$$

with

$$f_2 = \frac{1}{5} (z - 6) z^{1/5} \qquad \text{and} \qquad g_2 = z^{1/5} \,. \tag{38}$$

Using this explicit forms for F and Φ it is easy to see that $F \wedge \Phi$ is proportional to the quantity

$$g_1 f_2' + g_2 f_1' + g_1' (f_2 + g_2) + g_2' (f_1 + g_1) \,, \tag{39}$$

which vanishes after using (36) and (38). In the same way it is possible to show that the 2-form ω vanishes. This is easily seen in the orthonormal basis $\hat{e}^I$ introduced in [17].

A superpotential in terms of the calibration describing compactifications on Spin(7)-holonomy manifolds has been conjectured in [12]. It would be interesting to see if the conditions for unbroken supersymmetry obtained in this paper can actually be derived from the superpotential presented in [12].

Acknowledgments

I thank M. Becker, S. Gukov, J. Polchinski and E. Witten for useful discussions and correspondence. This work was supported by the U.S. Department of Energy under grant DE-FG03-92-ER40701.

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JHEP05(2001)003

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JHEP05(2001)003